Some Useful Formulae for CP Violation at BaBar

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The Physics Book provides an excellent reference for many useful formulae, but sometimes it is hard to find exactly the form needed, with all conventions defined. This note is supposed to be useful, not new. Perhaps it will be of some pedagogical value.

1 Oscillations

We write the Schrö dinger Equation for the coupled B^0 - $\overline{B}^0$ system as

$$i\frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} M - i\Gamma/2 & M_{12} \\ M_{12}^* & M - i\Gamma/2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \tag{1}$$

where a is the coefficient of $|B^0\rangle$ and b is the coefficient of $|\overline{B}^0\rangle$ We have ignored Γ_{12} as is appropriate for B_d (but not for B_s) because the common states to which both B^0 and $\overline{B}^0$ can both decay are Cabbibo suppressed. The (complex) eigenvalues of the matrix are

$$\mu_\pm = M - i\Gamma/2 \pm |M_{12}| \tag{2}$$

It is traditional to define the mass eigenstates as

$$\begin{aligned} |B_L\rangle &= p|B^0\rangle + q|\overline{B}^0\rangle \\ |B_H\rangle &= p|B^0\rangle - q|\overline{B}^0\rangle \end{aligned} \tag{3}$$

which can be inverted to get the (time-independent) states

$$\begin{aligned} |B^0\rangle &= \frac{1}{2p}[|B_L\rangle + |B_H\rangle] \\ |\overline{B}^0\rangle &= \frac{1}{2q}[|B_L\rangle - |B_H\rangle] \end{aligned} \tag{4}$$

A state that begins as B^0 at $t = 0$ evolves as

$$\begin{aligned} |B^0{}_{phys}(t)\rangle &= \frac{1}{2p}[e^{-i\mu_- t}(p|B^0\rangle + q|\overline{B}^0\rangle)e^{-i\mu_+ t}(p|B^0\rangle - q|\overline{B}^0\rangle)] \\ &= e^{-i(M-i\Gamma/2)t}[\cos(\Delta m t/2)|B^0\rangle + i\frac{q}{p}\sin(\Delta m t/2)|\overline{B}^0\rangle] \end{aligned} \tag{5}$$

and similarly

$$|\overline{B}^0{}_{phys}(t)\rangle \;=\; e^{-i(M-i\Gamma/2)t}[\cos(\Delta mt/2)|\overline{B}^0\rangle + i\frac{p}{q}\sin(\Delta mt/2)|B^0\rangle] \tag{6}$$

For each of the eigenvalues we can determine the eigenvector. Following the convention above for the lighter eigenstate

$$\begin{aligned} (M - i\Gamma/2)p + M_{12}q &= (M - i\Gamma/2 - |M_{12}|)p \\ q/p &= -\frac{|M_{12}|}{M_{12}} = -\frac{M_{12}^*}{|M_{12}|} \end{aligned} \tag{7}$$

where that $-$ sign is the consequence of using the lighter state to define p and q.

2 CP Violation from Mixing

Now consider the decay to a state f that is an eigenstate of CP:

$$CP|f\rangle = \eta_f|f\rangle \tag{8}$$

Now let us define the decay amplitudes from B^0 and $\overline{B}^0$ to f:

$$\begin{aligned} A &= \langle f|\mathcal{H}|B^0\rangle \\ \overline{A} &= \langle f|\mathcal{H}|\overline{B}^0\rangle \end{aligned} \tag{9}$$

The central quantity in our considerations is

$$\lambda = \frac{q}{p}\frac{\overline{A}}{A} \tag{10}$$

Note that both factors here depend upon phase conventions, which have not been specified yet. We return to this below.

The decay rates are proportional to the absolute squares of the amplitudes. We suppress the exponential $e^{-\Gamma t}$ common throughout.

$$
\begin{aligned}
|\langle f|\mathcal{H}|{B^0}_{phys}(t)\rangle|^2 &= |A|^2|\cos(\Delta mt/2) + i\lambda\sin(\Delta mt/2)|^2 \\
&= |A|^2|[\tfrac{1}{2}(1+\cos\Delta mt) + \tfrac{1}{2}|\lambda|^2(1-\cos\Delta mt) \\
&\qquad +2\Re i\lambda\sin(\Delta mt/2)\cos(\Delta mt/2)] \\
&= |A|^2[\tfrac{1}{2}(1+|\lambda|^2) + \tfrac{1}{2}(1-|\lambda|^2)\cos\Delta mt - \Im\lambda\sin\Delta mt] \\
|\langle f|\mathcal{H}|{\overline{B}^0}_{phys}(t)\rangle|^2 &= |\overline{A}|^2[\tfrac{1}{2}(1+\cos\Delta mt) + \tfrac{1}{2}|\frac{1}{\lambda|^2}(1-\cos\Delta mt) - \Im\frac{1}{\lambda}\sin\Delta mt] \\
&= |A|^2[\tfrac{1}{2}(1+|\lambda|^2) - \tfrac{1}{2}(1-|\lambda|^2)\cos\Delta mt + \Im\lambda\sin\Delta mt]
\end{aligned} \tag{11}
$$

This result agrees with that of Helen Quinn and Tony Sanda in RPP 2000, pp. 627-8. Note that going from B^0 to $\overline{B}^0$ the coefficients of both $\sin\Delta mt$ and $\cos\Delta mt$ change.

If we write λ out explicitly we find

$$
\begin{aligned}
\lambda &= \frac{q}{p}\frac{\overline{A}}{A} = -\frac{|M_{12}|}{M_{12}}\frac{\overline{A}}{A} \\
&= -\frac{|\langle B^0|\mathcal{H}|\overline{B}^0\rangle|}{\langle B^0|\mathcal{H}|\overline{B}^0\rangle}\frac{\langle f|\mathcal{H}|\overline{B}^0\rangle}{\langle f|\mathcal{H}|B^0\rangle}
\end{aligned} \tag{12}
$$

This makes explicit the independence of λ on our choice of phases for either B^0 or $\overline{B}^0$. Redefining either by, say $|B^0\rangle \to e^{i\zeta}|B^0\rangle$, leaves λ unchanged.

Now the weak Hamiltonian contains many pieces: strangeness raising, b-ness raising trees, b-ness raising penguins, ... strangeness lowering, b-ness lowering trees, etc. Were there no weak phases from the CKM matrix, the theory would be CP conserving. The full weak Hamiltonian would be

$$\mathcal{H} = \sum \mathcal{H}_j + \sum \mathcal{H}_j^\dagger \tag{13}$$

where

$$CP\mathcal{H}_jCP = \mathcal{H}_j^\dagger \tag{14}$$

When there are weak phases present we have instead

$$\mathcal{H} = \sum e^{i\phi_j}\mathcal{H}_j + \sum e^{-i\phi_j}\mathcal{H}_j^\dagger \tag{15}$$

The phases are opposite for the two pieces because the Hamiltonian must be equal to its own adjoint so that the theory will be unitary (this isn't exactly the same sense of unitary as in the unitary triangle, but almost).

Suppose that only a single piece, $\mathcal{H}_k$, contributes to $B^0 \to f$. Then only $\mathcal{H}_k^\dagger$, contributes to $\overline{B}^0 \to f$. From this we conclude that

$$\begin{aligned} A = \langle f|\mathcal{H}|B^0\rangle &= \langle f|e^{i\phi_k}\mathcal{H}_k|B^0\rangle \\ \overline{A} = \langle f|\mathcal{H}|\overline{B}^0\rangle &= \langle f|e^{-i\phi_k}\mathcal{H}_k^\dagger|\overline{B}^0\rangle \\ &= \langle f|e^{-i\phi_k}CP\mathcal{H}_k CP|\overline{B}^0\rangle \\ &= e^{-2i\phi_k}\eta_f\langle f|e^{i\phi_k}\mathcal{H}_k|B^0\rangle\langle B^0|CP|\overline{B}^0\rangle \\ &= e^{-2i\phi_k}\eta_f A\langle B^0|CP|\overline{B}^0\rangle \end{aligned} \tag{16}$$

Now we can write λ as

$$\lambda = -\frac{|\langle B^0|\mathcal{H}|\overline{B}^0\rangle|}{\langle B^0|\mathcal{H}|\overline{B}^0\rangle}\eta_f e^{-2i\phi_k}\langle B^0|CP|\overline{B}^0\rangle \tag{17}$$

Again the independence of phase convention is manifest.

Now we almost know the phase introduced by mixing. The quantity $\langle B^0|\mathcal{H}|\overline{B}^0\rangle$ is determined in the Standard Model by the box diagram with t quark intermediate states. Our matrix element has an outgoing B^0, i.e. an outgoing $\bar{b}$ quark. The transitions are thus $b \to t \to d, \overline{d} \to \bar{t} \to \bar{b}$. Each of these introduces $V_{tb}V_{td}^*$, so the phase is given by $V_{td}^{*2} \propto e^{2i\beta}$. This goes in the denominator of λ, so λ has the phase $e^{-2i\beta}$.

The problem is that we haven't determined the overall sign of the result! In fact, the matrix element is phase-convention dependent. It is only when it is combined with $\langle B^0|CP|\overline{B}^0\rangle$ that we get a well-defined result.

The classic calculation is that of T. Inami and C. S. Lim, *Prog. Theo. Phys.* **65**, 297 (1981). In the Physics Book, the result is cited as

$$M_{12} = -\frac{G_F^2}{12\pi^2}\eta_{QCD}m_B(B_B f_B^2)m_t^2 f_2(m_t^2/m_W^2)(V_{tb}V_{td}^*)^2\langle B^0|CP|\overline{B}^0\rangle \tag{18}$$

The factor η_{QCD} is a QCD correction and is a positive number. The function f_2 is a positive kinematic factor. B_B results from a hadronic matrix element of a four-quark operator:

$$\langle \overline{B}^0 | \bar{b}_\alpha \gamma_\mu (1-\gamma_5) d_\alpha \bar{b}_\beta \gamma^\mu (1-\gamma_5) d_\beta | B^0 \rangle = B_B (8/3) M_B^2 f_B^2 \tag{19}$$

where f_B is the (leptonic) B decay constant analogous to f_π. The (8/3) is the naive result obtained by inserting a vacuum intermediate state in the matrix element and contracting in all (four) possible ways. Color mismatching leads to the funny fraction. Typical lattice gauge calculations give $B_B = 1.3 \pm 0.1$ (e.g. J. M. Flynn in ICHEP 1996, p. 335.) Altogether, M_{12} cancels the phase dependence and introduces a minus sign, leaving us

$$\lambda = e^{-2i\beta - 2i\phi_k} \eta_f \tag{20}$$

where ϕ_k is the weak phase of the decay $B \to f$, assuming that only amplitude contributes.

The decay $B^0 \to \psi K_S$ is given by $b \to c\bar{c}s$. All these quarks are from the second and third generation. In the Wolfenstein parameterization we note that phases occur only for transitions between the first and third generations. Even the penguin diagram for $B^0 \to \psi K_S$ has no phase.

Thus λ is given by

$$\lambda = -e^{-2i\beta} \tag{21}$$

The minus sign is from η_f. (The ψ has spin one, the K_S has no spin. They need a p-wave to make the B^0, which introduces one factor of -1. Intrinsically, the ψ has $P = -1, C = -1, CP = +1$ while the K_S is $CP = +1$ (mostly). Hence $\eta_f = -1$.

Altogether,

$$\Im\lambda = \sin 2\beta; \qquad (B \to \psi K_S) \tag{22}$$

as stated in the Physics Book, Eq. 1.123, p. 32.

For the case of $B \to \pi\pi$, if we consider only tree diagrams, the operative transition for B^0 decay is $\bar{b} \to \bar{u}u\bar{d}$. This introduces V_{ub}^* whose phase is $+\gamma$. Remember that ϕ_k is the phase in B^0 decay so $\phi_k = \gamma$. Here $\eta_f = +1$, since there is no spin to worry about, just pseudoscalars. Thus

$$\lambda = e^{-2i\beta - 2i\gamma} = e^{2i\alpha} \tag{23}$$

where we assumed $\alpha + \beta + \gamma = \pi$. With this assumption

$$\Im\lambda = \sin 2\alpha; \qquad (B \to \pi\pi) \tag{24}$$

3 Non CP eigenstates

Eq.(**??**) is perfectly general, independent of the nature of the final state, f. To consider non-CP eigenstates we follow the physics book writing

$$\begin{aligned} A_f &= \langle f|\mathcal{H}|B^0\rangle \\ \overline{A}_f &= \langle f|\mathcal{H}|\overline{B}^0\rangle \\ A_{\overline{f}} &= \langle \overline{f}|\mathcal{H}|B^0\rangle \\ \overline{A}_{\overline{f}} &= \langle \overline{f}|\mathcal{H}|\overline{B}^0\rangle \end{aligned} \tag{25}$$

and

$$\begin{aligned} \lambda_f &= \frac{q}{p}\frac{\overline{A}_f}{A_f} \\ \lambda_{\overline{f}} &= \frac{q}{p}\frac{\overline{A}_{\overline{f}}}{A_{\overline{f}}} \end{aligned} \tag{26}$$

Now we specialize to the case where there is only one weak decay mechanism present, or at least all weak decay mechanisms have the same phase. Then we can write, with $\mathcal{A}$ and $\overline{\mathcal{A}}$ real

$$\begin{aligned} A_f &= \mathcal{A}e^{i\delta_f}e^{i\phi_f} \\ \overline{A}_f &= \overline{\mathcal{A}}e^{i\overline{\delta}_f}e^{-i\overline{\phi}_f} \\ A_{\overline{f}} &= \overline{\mathcal{A}}e^{i\overline{\delta}_f}e^{i\overline{\phi}_f}\langle\overline{B}^0|CP|B^0\rangle \\ \overline{A}_{\overline{f}} &= \mathcal{A}e^{i\delta_f}e^{-i\phi_f}\langle B^0|CP|\overline{B}^0\rangle \end{aligned} \tag{27}$$

We have, therefore

$$\begin{aligned} \lambda_f &= \frac{q}{p}\frac{\overline{A}_f}{A_f} = \frac{q}{p}\frac{\overline{\mathcal{A}}}{\mathcal{A}}e^{i(\delta-2\phi)} \\ \lambda_{\overline{f}} &= \frac{q}{p}\frac{\overline{A}_{\overline{f}}}{A_{\overline{f}}} = \frac{q}{p}\frac{\mathcal{A}}{\overline{\overline{\mathcal{A}}}}e^{i(-\delta-2\phi)} \end{aligned} \tag{28}$$

where

$$\delta = \overline{\delta}_f - \delta_f; \qquad 2\phi = \overline{\phi}_f + \phi_f \tag{29}$$

We can now insert these relations into Eq.(**??**) using

$$\frac{q}{p} = -\frac{e^{-2i\beta}}{\langle B^0|CP|\overline{B}^0\rangle} \tag{30}$$

$$\begin{aligned}
\lambda_f &= -\frac{e^{-2i\beta}}{\langle B^0|CP|\overline{B}^0\rangle}\frac{\overline{\mathcal{A}}}{\mathcal{A}}e^{i(\delta-2\phi)} \\
\lambda_{\overline{f}} &= -\frac{e^{-2i\beta}}{\langle B^0|CP|\overline{B}^0\rangle}\frac{\mathcal{A}}{\overline{\overline{\mathcal{A}}}}e^{i(-\delta-2\phi)}\frac{\langle B^0|CP|\overline{B}^0\rangle}{\langle \overline{B}^0|CP|B^0\rangle} = -\frac{e^{-2i\beta}}{\langle \overline{B}^0|CP|B^0\rangle}\frac{\mathcal{A}}{\overline{\overline{\mathcal{A}}}}e^{i(-\delta-2\phi)}
\end{aligned} \tag{31}$$

Both λ_f and $\lambda_{\overline{f}}$ are independent of the phase conventions chosen for $|B^0\rangle$ and $\overline{B}^0$. Of course, we have specifically chosen the Wolfenstein parameterization for the CKM matrix.

We see that

$$\frac{1}{\langle B^0|CP|\overline{B}^0\rangle}\frac{\overline{\mathcal{A}}}{\mathcal{A}} = r \tag{32}$$

is real and

$$\frac{1}{\langle \overline{B}^0|CP|B^0\rangle}\frac{\mathcal{A}}{\overline{\overline{\mathcal{A}}}} = \frac{1}{r} \tag{33}$$

Collecting all these results,

$$\begin{aligned}
|\langle f|\mathcal{H}|B^0{}_{phys}(t)\rangle|^2 &= |\mathcal{A}|^2[\tfrac{1}{2}(1+r^2) + \tfrac{1}{2}(1-r^2)\cos\Delta mt - r\sin(2\beta+2\phi-\delta)\sin\Delta mt] \\
|\langle f|\mathcal{H}|\overline{B}^0{}_{phys}(t)\rangle|^2 &= |\mathcal{A}|^2[\tfrac{1}{2}(1+r^2) - \tfrac{1}{2}(1-r^2)\cos\Delta mt + r\sin(2\beta+2\phi-\delta)\sin\Delta mt] \\
|\langle \overline{f}|\mathcal{H}|B^0{}_{phys}(t)\rangle|^2 &= |\overline{\mathcal{A}}|^2[\tfrac{1}{2}(1+r^{-2}) + \tfrac{1}{2}(1-r^{-2})\cos\Delta mt - \frac{1}{r}\sin(2\beta+2\phi+\delta)\sin\Delta mt] \\
|\langle \overline{f}|\mathcal{H}|\overline{B}^0{}_{phys}(t)\rangle|^2 &= |\overline{\mathcal{A}}|^2[\tfrac{1}{2}(1+r^{-2}) - \tfrac{1}{2}(1-r^{-2})\cos\Delta mt + \frac{1}{r}\sin(2\beta+2\phi+\delta)\sin\Delta mt]
\end{aligned} \tag{34}$$

We can write the final two expressions as

$$\begin{aligned}
|\langle \overline{f}|\mathcal{H}|B^0{}_{phys}(t)\rangle|^2 &= |\mathcal{A}|^2[\tfrac{1}{2}(1+r^2) - \tfrac{1}{2}(1-r^2)\cos\Delta mt - r\sin(2\beta+2\phi+\delta)\sin\Delta mt] \\
|\langle \overline{f}|\mathcal{H}|\overline{B}^0{}_{phys}(t)\rangle|^2 &= |\mathcal{A}|^2[\tfrac{1}{2}(1+r^2) + \tfrac{1}{2}(1-r^2)\cos\Delta mt + r\sin(2\beta+2\phi+\delta)\sin\Delta mt]
\end{aligned} \tag{35}$$

Note that since we assumed that there was only one weak phase, if f is, in fact, a CP eigenstate, $r = \pm 1$.

This formalism is appropriate to final states like $\rho\pi$ and $D^* D$. In fact for charged final states this are transformed into each other by $c \leftrightarrow u$.